Non-Standard Infinite Sums

(Warning: not for answering questions
on the AP Calculus Exam)

(definitely for fans of Srinivasa Ramanujan,
Euler, Riemann Zeta Function,
String (M) theory, ...)

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Let $A = \sum_{n=0}^{\infty} (-1)^n$

$$A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots$$

$$A = (1-1) + (1-1) + (1-1) + (1-1) + \cdots = 0 + 0 + 0 + \cdots = 0$$

$$A = 1 + (-1+1) + (-1+1) + (-1+1) + \cdots = 1 + 0 + 0 + \cdots = 1$$

$$\begin{aligned} 1 - A &= 1 - (1 + 1 - 1 + \dots) \\ &= 1 + 1 - 1 + \cdots = A \end{aligned}$$

If $1 - A = A$ then $1 = 2A$, so $A = \frac{1}{2}$

Adding the counting numbers

Karl Friedrik Gauß (1777-1855)

$$
\begin{array}{r}
1 + \ \ 2 + \cdots + \ \ 99 + 100 \\
100 + \ 99 + \cdots + \ \ \ 2 + \ \ \ 1 \\
\hline
101 + 101 + \cdots + 101 + 101 = 100(101)
\end{array}
$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Ernesto Cesàro (1859-1906)

Cesàro Summation

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + 5 + \cdots = \quad -\frac{1}{12}$$

Cesàro Convergence

$$A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots$$

$$B = 1 - 2 + 3 - 4 + 5 - 6 + 7 + \dots$$

$$C = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \dots$$

$$\begin{aligned} 2B = & (1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots) \\ & +(\quad 1 - 2 + 3 - 4 + 5 - 6 + \dots) \\ \hline & = (1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots) = A \end{aligned}$$

$$\begin{aligned} C - B = & (1 + 2 + 3 + 4 + 5 + 6 + 7 + \dots) \\ & -(1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots) \\ \hline & = (0 + 4 + 0 + 8 + 0 + 12 + 0 + \dots) = 4C \end{aligned}$$

If $A = \frac{1}{2}$, then $B = \frac{1}{4}$

If $C - B = 4C$, then $-B = 3C$

so $C = \frac{1}{3}(-B) = \frac{1}{3}\left(-\frac{1}{4}\right) = \quad -\frac{1}{12}$